Relativistic electron in a quantised plane wave

1985 J. Phys. A: Math. Gen. 18 1675

(http://iopscience.iop.org/0305-4470/18/10/022)

J. Phys. A: Math. Gen. **18** (1985) 1675-1685. Printed in Great Britain

Relativistic electron in a quantised plane wave

Piotr Filipowicz†
Max-Planck-Institut für Quantenoptik, D-8046 Garching, West Germany

Received 10 January 1985

Abstract. We have found an orthogonal set of solutions of the Dirac equation for the electron interacting with a quantised electromagnetic plane wave. The orthogonality of the wavefunctions is proved and the physical interpretation of the solutions is discussed.

1. Introduction

The continuous development of strong radiation sources caused great interest in theoretical work on the interaction of charges with strong electromagnetic fields. A very important problem in quantum mechanics was to find exact solutions of wave equations for an electron interacting with an electromagnetic plane wave. Knowledge of these solutions is very useful as it enables us to evaluate the ranges of validity of perturbation methods. When the magnetic field is added this knowledge is necessary, since in this case the interaction can be resonant and thus the use of perturbation methods is rather difficult. Most existing solutions have been obtained under the assumption that the electromagnetic field is an external classical plane wave. Solutions for the quantised external wave have been obtained only recently (Berson 1969). One can obtain exact solutions only in the case of the external one-dimensional electromagnetic field (it is possible to include only those modes which propagate in the same direction). The solvability of problems of this type results from the fact that the wavevectors are null vectors and from the assumption that they are mutually orthogonal and also orthogonal to the electromagnetic field tensor (in the four-dimensional sense).

The Dirac equation for the electron interacting with the classical external electromagnetic plane wave has a long history. Volkov solved this equation in 1935 shortly after it was proposed by Dirac. In the sixties, because of the rapid development of laser physics, experimental verification of the effects which follow from the Volkov solutions became possible. This triggered renewed interest in the problem of the fast electron in strong external fields (Sengupta 1949, Brown and Kibble 1964, Redmond 1965). Chakrabarti (1968) found solutions in the case when the anomalous magnetic moment was taken into account. He has shown that one can obtain Volkov states which transform the free-electron solution by a certain unitary operator. Beers and Nickle (1972) found that it is possible to solve relativistic wave equations by the algebraic methods connected with the reduced Poincaré group, which does not change the electromagnetic field tensor of the one-mode plane wave. This method permits us to interpret the Volkov solutions by algebraic methods. Backer and Mitter (1974) used

† Permanent address: Institute of Physics, Polish Academy of Sciences, Aleja Lotnikow 32/46, PL-02-668 Warszawa, Poland.

the projection operator technique proposed by Neville and Röhrlich (1971a, b) to solve the Dirac equation. In connection with the possibility of using Volkov solutions as a starting point for perturbation methods, it is important to prove orthogonality of wavefunctions. Freid *et al* (1966) and Von Ross (1966) thought that the Dirac Hamiltonian is not Hermitian and that the Volkov states do not form an orthogonal set. Eberly (1969) and Ritus (1979) proved the orthogonality of wavefunctions for the Klein–Gordon equation and the Dirac equation respectively.

In the case of the quantised one-mode linearly polarised plane wave, the Dirac equation was first solved by Berson (1969) and Fedorov and Kozakov (1973). Their calculations were done in the Bargman representation of the field operators. Bialynicki-Birula (1980) found solutions in the phase representation of creation and annihilation operators. Bergou and Ehlotzky (1983) obtained wavefunctions independent of the field operators representation. The authors of these papers did not find a spin operator that commutes with the Dirac operator. Thus they did not obtain wavefunctions labelled by the eigenvalues of such a spin operator.

In this paper we propose solutions of the Dirac equation for an electron interacting with the circularly polarised one-mode electromagnetic field which are in a much simpler and more useful form than the existing solutions. It is possible to include other modes propagating in the same direction but the one-mode model is sufficiently representative. The wavefunctions obtained are the eigenfunctions of the Pauli–Lubanski spin operator (Lubanski 1942). We found, for the first time, that this operator commutes with a Dirac operator and also prove the orthogonality of wavefunctions for the quantised field. We have shown the connection between these solutions and the Volkov states in the limit of large photon numbers.

In all the references cited above the interpretation of the results is based on the notion of the generalised electron momentum. It is well known that such an interpretation leads in many cases to the wrong results. In this paper wavefunctions are interpreted by use of the expectation value of the kinetic momentum operator which is closely connected with measurements. The wavefunctions obtained for the quantised plane wave are not symmetric with respect to charge conjugation. For certain quantum numbers the momentum and energy of the positron continuously increase with time. This takes place even for the electromagnetic vacuum. This non-physical asymmetry could not be obtained using perturbation methods. It is possible to somewhat modify the wave equation to obtain a charge symmetric theory. We return to this problem at the end of the paper. The solutions presented are characterised by the simplicity of form. This reduces the calculations in perturbation theory in the Furry picture for an electron interacting with the quantised plane wave. Solutions can be used to find the non-linear dependence on the intensity of the wave of various effects such as Compton scattering, bremsstrahlung etc. They do not have great physical meaning for small quantum numbers of the field as all the modes but one are neglected. On the other hand, it is necessary to know the solutions for the quantised plane wave if the photon statistics is far from classical.

2. Volkov states

The solution of the Dirac equation for an electron interacting with electromagnetic plane wave was first given by Volkov (1935):

$$[\mathrm{i}\not{\partial} - e\not{A}(kx) - m]\psi(x) = 0, \tag{1}$$

where A is the electromagnetic potential and k the null vector of the plane wave. This famous solution has focused physicists' attention up to the present day (Brown and Kowalski 1983, Bergou and Varro 1980). We write it as (Landau and Lifshitz 1973)

$$\psi_{pr}(x) = e^{iS_p(x)}\phi_{pr}(kx), \tag{2}$$

where p is the generalised electron momentum, which is on the mass shell, r is the quantum spin number,

$$S_p(x) = -px - \frac{1}{2kp}\int_0^{kx} (2epA(\xi) - e^2A^2(\xi))\,d\xi \tag{3}$$

classical action function,

$$\phi_{pr}(kx) = [1 + (e/2kp)\boldsymbol{k}\boldsymbol{A}(kx)]u_{pr} \tag{4}$$

and the bispinor u_{pr} is the solution of the equation

$$(\boldsymbol{p} - m)u_{pr} = 0, \qquad r = 1, 2. \tag{5}$$

We have used the following notation in the expression written above. The scalar product of two 4-vectors is defined as $ab = a_\mu b^\mu = g_{\mu\nu}a^\mu b^\nu$. The metric tensor is chosen in the $(+---)$ sign convention. The natural $h/2\pi = c = 1$ relativistic units are used. The γa scalar product is denoted by $\boldsymbol{a}$. We omit the scalar multiplication sign in all expressions.

We will now show explicitly the method of obtaining the Volkov solution. To do this one should solve the Dirac equation (1). There are three operators which commute with the Dirac operator, the transverse part of the kinetic momentum $i\partial_{tr}$ and the projection of the kinetic momentum on the wavevector $ik\partial$. The eigenfunction of these operators has the form

$$\psi_p(x) = \exp(-ipx)\phi_p(kx), \tag{6}$$

the vector p is arbitrary, the eigenvalues are: p_{tr}, kp; the function ϕ_p is arbitrary. There is some freedom in the choice of vector p. It can be shifted by any vector proportional to the wavevector k without changing the eigenvalues. The resulting factor $\exp(-ikx)$ can be absorbed in ϕ, therefore vector p can be fixed on the mass shell $p^2 = m^2$.

Solutions of the Dirac equation can be assumed to be in the form (6). The function ϕ_p should satisfy the first-order differential equation

$$(\boldsymbol{p} + i\boldsymbol{k}\, d/d\tau - e\boldsymbol{A}(\tau) - m)\phi_p(\tau) = 0. \tag{7}$$

This equation can be solved if one multiplies it from the left-hand side by matrix $\boldsymbol{k}$. The following identity is obtained

$$\phi_p(\tau) = (1/2kp)(\boldsymbol{p} - e\boldsymbol{A}(\tau) + m)\boldsymbol{k}\phi_p(\tau). \tag{8}$$

If we substitute (8) into (7) we obtain a simpler equation for $\phi_p(\tau)$:

$$i(d/d\tau)\phi_p(\tau) = (1/2kp)(-2epA(\tau) + e^2A^2(\tau))\phi_p(\tau), \tag{9}$$

in which the components of the bispinor are decoupled. If we integrate (9) we obtain the solution in a more general form than that stated by Volkov (2):

$$\psi_p(x) = \exp(iS_p(x))\phi_p(kx), \tag{10}$$

where

$$\phi_p(kx) = (1/2kp)(\boldsymbol{p} - e\boldsymbol{A}(kx) + m)\boldsymbol{k}u_p$$

and u_p is an arbitrary bispinor. The Volkov solution is a special case of (10) for u_p satisfying an additional condition

$$(\boldsymbol{p}-m)u_p=0. \tag{11}$$

In this paper we restrict ourselves to the case of the one-mode circularly polarised plane wave. This field is described by the following vector potential

$$A_\mu(x)=\lambda(\varepsilon\exp(-\mathrm{i}kx))+\varepsilon^*\exp(\mathrm{i}kx)),$$

where ε is a polarisation vector. We prefer the form (10) for the following reasons. Firstly the proof of orthogonality is much simpler in this form rather than the one proposed by Ritus (1979). Secondly no one has found a spin operator which commutes with Dirac operator for which the function (2) is an eigenfunction. It turns out that the function of (10) is an eigenfunction of the projection of the Pauli–Lubanski spin operator (Lubanski 1942)

$$W_{\mu\nu}=\tfrac{1}{2}\varepsilon^{\mu\nu\delta\lambda}\sigma_{\nu\delta}(\mathrm{i}\partial_\lambda-eA_\lambda(kx)),\qquad \sigma_{\nu\delta}=\mathrm{i}\tfrac{1}{2}[\gamma_\nu,\gamma_\delta] \tag{12}$$

on the wavevector k if we impose a certain condition on bispinor u_p. The condition is that u_p should be an eigenfunction of the matrix

$$\boldsymbol{S}=\tfrac{1}{2}[\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^*]. \tag{13}$$

with eigenvalues $s=\pm1$. We note that the matrix $\boldsymbol{S}$ does not commute with matrix $\boldsymbol{p}-m$.

We present now a proof of orthogonality of functions (10) on the hyperplane $x_0=$ constant, i.e. we show that

$$\langle p'|p\rangle=\int_{x_0=\text{constant}}\bar{\psi}_{p'}(x)\gamma_0\psi_p(x)=(2\pi)^3\delta(\boldsymbol{p}'-\boldsymbol{p})(p_0/2kp)u_p\boldsymbol{k}u_p. \tag{14}$$

This integral can be explicitly performed over the spatial variables x_1 and x_2. Then we obtain

$$\langle p'|p\rangle=(2\pi)^2\delta(p_1'-p_1)\delta(p_2'-p_2)\int_{-\infty}^{\infty}\exp\left[\mathrm{i}\tfrac{1}{2}(np'-np)\left((x_0+x_3)-\frac{\omega}{(kp')(kp)}\right.\right.$$
$$\left.\left.\times\int_0^{kx}[m^2+(\boldsymbol{p}_{\rm tr}-e\boldsymbol{A}(\xi))^2]\,\mathrm{d}\xi\right)\right]\bar{\phi}_{p'}\gamma_0\phi_p|_{\boldsymbol{p}'_{\rm tr}=\boldsymbol{p}_{\rm tr}}\,\mathrm{d}x_3,$$

where

$$\bar{\phi}_{p'}\gamma_0\phi_p|_{\boldsymbol{p}'_{\rm tr}=\boldsymbol{p}_{\rm tr}}=\frac{1}{2\omega}\left(1+\frac{\omega^2}{(kp')(kp)}[m^2+(\boldsymbol{p}_{\rm tr}-e\boldsymbol{A}(kx))^2]\right)\bar{u}_{p'}\boldsymbol{k}u_p|_{\boldsymbol{p}'_{\rm tr}=\boldsymbol{p}_{\rm tr}},$$
$$k=\omega n=\omega(1,0,0,1). \tag{15}$$

We introduce a new integration variable

$$u=x_3-\frac{\omega}{(kp')(kp)}\int_0^{kx}[m^2+(\boldsymbol{p}_{\rm tr}-e\boldsymbol{A}(\xi))^2]\,\mathrm{d}\xi. \tag{16}$$

The integral (15) becomes

$$\langle p'|p\rangle=(2\pi)^3\delta(p_1'-p_1)\delta(p_2'-p_2)\delta(kp'-kp)\bar{u}_p\boldsymbol{n}u_p. \tag{17}$$

Changing kp variable in the δ function we finally obtain (14).

We note that the solution of the Klein-Gordon equation

$$[(\mathrm{i}\partial - eA(kx))^2 - m^2]\psi(x) = 0 \tag{18}$$

is much simpler. Substituting expression (6) to equation (18) we find that

$$\psi_p(x) = \exp(\mathrm{i}S_p(x)) \tag{19}$$

is a solution. Wavefunctions (19) are orthogonal on the hyperplane $x_0 = \text{constant}$ (Eberly 1969)

$$\langle p'|p\rangle = \int_{x_0=\text{constant}} \mathrm{d}^3x\, \psi^*_{p'}(x)\mathrm{i}\overleftrightarrow{\partial}_0\, \psi_p(x) = (2\pi)^3\delta(\boldsymbol{p}' - \boldsymbol{p})2p_0, \tag{20}$$

where $f\overleftrightarrow{\partial}_0 g$ has the meaning: $f\overleftrightarrow{\partial}_0 g = f(\partial_0 g) - (\partial_0 f)g$.

3. Quantised one-mode field

In this section we consider the interaction of an electron with circularly polarised one-mode electromagnetic field. Such a field is described by electromagnetic vector potential A of the form

$$A_\mu(kx) = g(\varepsilon_\mu a \exp(-\mathrm{i}kx) + \varepsilon^*_\mu a^\dagger \exp(\mathrm{i}kx)), \tag{21}$$

where a, $a^\dagger$ are boson operators, k the wavevector, g the coupling constant, ε the polarisation vector with the following properties

$$\varepsilon^*\varepsilon = -1, \qquad \varepsilon\varepsilon = \varepsilon^*\varepsilon^* = 0 \quad (\varepsilon^* \text{ is a complex conjugate to } \varepsilon).$$

The Dirac equation for such a field can be solved (Filipowicz 1980) in the form of a plane wave if one notices that the coordinate dependence of the potential $A(kx)$ can be removed by the unitary operator

$$U = \exp(-\mathrm{i}kxN) \tag{22}$$

where N is the occupation number operator: $N = \frac{1}{2}(a^\dagger a + aa^\dagger)$. It follows that the transformed function $\phi(x) = U\psi(x)$ satisfies the equation without explicit coordinate dependence

$$(\mathrm{i}\boldsymbol{\partial} - \boldsymbol{k}N - e\boldsymbol{A} - m)\phi(x) = 0. \tag{23}$$

Here $\boldsymbol{A} = g(\boldsymbol{\varepsilon}a + \boldsymbol{\varepsilon}^* a^\dagger)$. One can find the solution of (23) in the plane wave form

$$\phi(x) = \exp(-\mathrm{i}px)\phi. \tag{24}$$

The bispinor ϕ satisfies the matrix equation with coefficients depending on creation and annihilation operators

$$(\boldsymbol{p} - \boldsymbol{k}N - e\boldsymbol{A} - m) = 0. \tag{25}$$

If we multiply (25) from the left-hand side by matrix $\boldsymbol{k}$ we get the following identity

$$\phi = (1/2kp)(\boldsymbol{p} - e\boldsymbol{A} + m)\boldsymbol{k}\phi. \tag{26}$$

Substituting (26) into equation (25) we obtain a simpler equation for the bispinor ϕ:

$$[(p - eA)^2 - 2kpN - e^2g^2\boldsymbol{S} - m^2]\boldsymbol{k}\phi = 0, \tag{27}$$

where $S = \frac{1}{2}[\varepsilon, \varepsilon^*]$ is a spin operator. Now we introduce two projection operators

$$Q_+ = \tfrac{1}{2}k'k, \qquad Q_- = \tfrac{1}{2}kk', \tag{28}$$

where k' is the null vector orthogonal to the wavevector k. These operators have the following properties: $Q_\pm Q_\pm = Q_\pm$, $Q_\pm Q_\mp = 0$, $Q_+ + Q_- = 1$. It is easy to show that

$$k\phi = k\phi_+, \qquad \text{where } \phi_+ = Q_+\phi. \tag{29}$$

Equation (27) is in fact an equation for projection ϕ_+ and the identity (26) reproduces the function ϕ from ϕ_+. In order to satisfy (27) the projection ϕ_+ should be the direct product of the field and spin states:

$$\phi_+ = \chi_s|\xi\rangle, \tag{30}$$

where $|\xi\rangle$ is a state depending on the field operators only. Moreover χ_s should be an eigenfunction of the spin operator S:

$$S\chi_s = s\chi_s, \qquad s = \pm 1$$

with the auxiliary condition $Q_-\chi_s = 0$. We notice that the spin operator S commutes with matrices k and k'. After substituting (30) into (27) we obtain an equation for $|\xi\rangle$ which has the bilinear form of creation and annihilation operators acting on $|\xi\rangle$ to give zero. Such a bilinear form can be diagonalised with the help of a displacement unitary operator:

$$D_p = \exp(-\alpha_p a^\dagger + \alpha_p^* a), \tag{31}$$

where

$$\alpha_p = -[eg/(e^2g^2 + kp)](p\varepsilon^*).$$

The operator D_p shifts operators a and $a^\dagger$

$$a \to D_p a D_p^\dagger = a + \alpha_p. \tag{32}$$

After performing this transformation we obtain

$$\left(p^2 - 2kpN - 2e^2g^2(N + \tfrac{1}{2}s) + 2e^2g^2\frac{(p\varepsilon)(p\varepsilon^*)}{kp} + e^2g^2 - m^2\right)D_p|\xi\rangle = 0. \tag{33}$$

The function $D_p|\xi\rangle$ should be the eigenfunction of the occupation number operator

$$|\xi\rangle = D_p^\dagger|n\rangle. \tag{34}$$

After performing inverse transformations one finally obtains the solution of the Dirac equation

$$\psi_{psn}(x) = \exp(-\mathrm{i}px + \mathrm{i}kxN)\frac{1}{2kp}(p - eA + m)D_p^\dagger k\chi_s|n\rangle, \tag{35}$$

where vector p satisfies the following condition

$$p^2 - 2kp(n + \tfrac{1}{2}) - 2e^2g^2(n + \tfrac{1}{2} + \tfrac{1}{2}s) + 2e^2g^2\frac{(p\varepsilon)(p\varepsilon^*)}{kp + e^2g^2} - m^2 = 0. \tag{36}$$

The wavefunction (35) is an eigenfunction of the projection of the Pauli–Lubanski spin operator W on the wavevector k

$$kW\psi_{psn}(x) = skp\psi_{psn}(x). \tag{37}$$

Operator kW commutes with the Dirac operator.

Vector p does not lie on the mass shell. It is convenient to label solutions (35) with a different vector P which is on the mass shell and is connected with vector p by the formula

$$P = p - C_{ns}(kp, p\varepsilon, p\varepsilon^*), \tag{38}$$

where

$$C_{ns} = n + \tfrac{1}{2} + \frac{e^2g^2}{kp}(n + \tfrac{1}{2} + \tfrac{1}{2}s) - \frac{e^2g^2}{kp}\frac{(p\varepsilon)(p\varepsilon^*)}{kp + e^2g^2}. \tag{39}$$

This change of variables has the interesting property that the transformation (38) can be easily inverted:

$$p = P + C_{ns}(kP, P\varepsilon, P\varepsilon^*)k. \tag{40}$$

The wavefunctions (35) can be used in the calculation of various effects appearing in strong electromagnetic field. Thus their orthogonality is very important. We will show that

$$\begin{aligned}\langle P's'n'|Psn\rangle &= \int_{x_0=\text{constant}} \mathrm{d}^3x\, \bar{\psi}_{P's'n'}(x)\gamma_0\psi_{Psn}(x)\\ &= (P_0/kP)(\bar{\chi}_s \boldsymbol{k}\chi_s)(2\pi)^3\delta(\boldsymbol{P}' - \boldsymbol{P})\delta_{s's}\delta_{n'n}.\end{aligned} \tag{41}$$

From the continuity equation

$$\partial_\mu(\bar{\psi}_{P's'n'}(x)\gamma^\mu\psi_{Psn}(x)) = 0 \tag{42}$$

it follows that

$$(p_0' - p_0)\int \mathrm{d}^3x\, \bar{\psi}_{P's'n'}\gamma_0\psi_{Psn} = 0. \tag{43}$$

Integral (41) is equal to zero if $p_0' \neq p_0$, thus we can assume that $p_0' = p_0$. For this case we obtain

$$\begin{aligned}\langle P's'n'|Psn\rangle &= (2\pi)^3\delta(\boldsymbol{p}' - \boldsymbol{p})\langle n'|D_p\chi_{s'}\boldsymbol{k}(\boldsymbol{p} - e\boldsymbol{A} + m)\gamma_0(\boldsymbol{p} - e\boldsymbol{A} + m)\boldsymbol{k}\chi_s D_p^\dagger|n\rangle\frac{1}{4(kp)^2}\\ &= (2\pi)^3\delta(\boldsymbol{p}' - \boldsymbol{p})\delta_{n'n}\delta_{s's}(\chi_s\boldsymbol{k}\chi_s)(p_0/kp)M,\\ M &= 1 + \frac{k_0}{p_0}\left[\frac{e^2g^2}{kp}(n + \tfrac{1}{2} + \tfrac{1}{2}s) - e^2g^2\frac{(p\varepsilon)(p\varepsilon^*)(e^2g^2 + 2kp)}{kp(kp + e^2g^2)^2}\right).\end{aligned} \tag{44}$$

We will change variables inside the δ function

$$\delta(\boldsymbol{p}' - \boldsymbol{p}) = (1/M)\delta(\boldsymbol{P}' - \boldsymbol{P}). \tag{45}$$

It can be noticed that the complicated factor M in equations (44) and (45) cancels. The remaining terms give (41).

We write the wavefunction in a slightly different form:

$$\begin{aligned}\psi_{Psn}(x) &= \exp\left(-\mathrm{i}px + \mathrm{i}\frac{e^2g^2}{kP}\frac{(P\varepsilon)(P\varepsilon^*)}{kp + e^2g^2}\right)\frac{1}{2kP}(\boldsymbol{P} - e\boldsymbol{A}(kx) + m)\\ &\quad\times\exp\left(\mathrm{i}e\frac{PA'(kx)}{kP + e^2g^2}\right)\exp\left(-\mathrm{i}kx(n + \tfrac{1}{2}s)\right)\frac{e^2g^2}{kP}\chi_s|n\rangle.\end{aligned} \tag{46}$$

If we go to the limit of large occupation number of the field ($n \to \infty$) in such a way that $g\sqrt{n} = \lambda/\sqrt{2}$ remains constant, the wavefunction (46) will go to the solution of the Dirac equation for a classical field of the form

$$A(kx) = \lambda[\varepsilon \exp(-\mathrm{i}kx) + \varepsilon^* \exp(\mathrm{i}kx)].$$

The Klein-Gordon equation can be solved in a similar way. For this case we get the following wavefunction

$$\psi_{Pn}(x) = \exp\left(-\mathrm{i}Px + \mathrm{i}\frac{e^2g^2(P\varepsilon)(P\varepsilon^*)kx + ePA'(kx)}{kP + e^2g^2}\right) \exp\left(-\mathrm{i}kxN\frac{e^2g^2}{kP}\right)|n\rangle \tag{47}$$

where vector P is on the mass shell $P^2 = m^2$. These functions are orthogonal on the hyperplane $x_0 = \text{constant}$

$$\langle P'n'|Pn\rangle = \int_{x_0=\text{constant}} \mathrm{d}^3x\, \psi_{P'n'}(x)\mathrm{i}\overleftrightarrow{\partial}_0\psi_{Pn}(x) = (2\pi)^3 2P_0\delta(\boldsymbol{P}' - \boldsymbol{P}). \tag{48}$$

4. Physical interpretation

Wavefunctions (10) are labelled by parameters p^μ. These parameters satisfy the condition $p^2 = m^2$ but they do not form the kinetic momentum of the particle. It is necessary to find the connection between kinetic momentum and vector p. In order to do so we have to calculate the expectation value of the kinetic momentum operator in a certain state. This expectation value should be connected with the classical expression (Bieresteckij *et al* 1968) for the kinetic momentum.

We localise the particle in a region much smaller than the wavelength of the plane wave, $\Delta zk_z \ll 1$. This region should also be much larger than the Compton wavelength in order to avoid difficulties connected with interference of states with positive and negative frequencies. It is very difficult to use such a wavepacket in calculations of expectation values at a fixed time. It becomes simpler if one changes the variables from (t, z) to $(t-z, t+z)$. Localisation of particle at a fixed time is equivalent to localisation at a fixed value of kx (Neville 1971). We will find the expectation value of the kinetic momentum on the hyperplane $kx = \text{constant}$ in a state described by a wavepacket localised around certain $k'x$ values (Chakrabarti 1969)

$$\psi_{\mathrm{h}}(x) = \int \frac{\mathrm{d}^3\hat{p}}{(2\pi)^3} h(\hat{p})\psi_p(x), \tag{49}$$

where $\mathrm{d}^3\hat{p} = \mathrm{d}p_x\, \mathrm{d}p_y\, \mathrm{d}(kp)$, $\psi_p(x)$ is a solution of the Dirac equation for the classical field and $h(\hat{p})$ is a profile of wavepacket normalised in the following way

$$\int \frac{\mathrm{d}^3\hat{p}}{(2\pi)^3}|h(\hat{p})|^2 = 1.$$

For this state we obtain

$$\begin{aligned}\langle \pi^\mu(x)\rangle = \langle \mathrm{i}\partial^\mu - eA^\mu(kx)\rangle &= \int_{kx=\text{constant}} \mathrm{d}^3\hat{x}\, \bar{\psi}_{\mathrm{h}}(x)\boldsymbol{k}(\mathrm{i}\partial^\mu - eA(kx))\psi_{\mathrm{h}}(x) \\ &= \int \frac{\mathrm{d}^3\hat{p}}{(2\pi)^3}|h(\hat{p})|^2\left[(p^\mu - eA_\mu(kx)) - \frac{k_\mu}{2kp}(-2epA(kx) + e^2A^2(kx))\right]\frac{kp}{m}\bar{u}_p u_p\end{aligned}$$

where

$$d^3\hat{x} = dx\, dy\, d(k'x). \tag{50}$$

Bispinors u_p are normalised to one particle in the volume: $\bar{u}_p u_p = m/2kp$. Comparing this expression with the classical formula for the kinetic momentum (Bieresteckij 1968) we can make the following identification. The mean values $\int (d^3\hat{p}/(2\pi)^3)|h(\hat{p})|^2 p^\mu$ are the quantum counterparts of the classical parameters p^μ for sufficiently narrow wavepackets, $(m/kp)kx$ should be identified with the proper time. It can be shown that vector p is the momentum of the particle if the electromagnetic field is switched adiabatically. The square of the expectation value of the kinetic momentum averaged over time is equal to

$$\overline{\langle \pi(x)\rangle^2} = m_*^2 = m^2 + e^2\lambda^2, \tag{51}$$

m_* is called the effective mass of the particle interacting with the plane wave.

It is very interesting to find the expectation value of the kinetic momentum in the case of quantised field and show the relation with the semiclassical expression (50). Solutions (35) are additionally labelled by the quantum number n, which describes the field state. In order to elucidate semiclassical correspondence we will assume that the electromagnetic field is in a coherent state. In the present case the wavefunction has the form

$$\psi_{hs\alpha}(x) = \int \frac{d^3\hat{p}}{(2\pi)^3} h(\hat{p}) \sum_{n=0}^{\infty} \exp - \frac{|\alpha|^2}{2} \frac{\alpha^n}{\sqrt{n!}} \psi_{psn}(x), \tag{52}$$

where ψ_{psn} is a solution of (35) and coherent state

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

is the eigenstate of annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$. After performing the necessary calculations we find that the expectation value of the kinetic momentum on the hyperplane $kx = $ constant is equal to

$$\begin{aligned}\langle \pi^\mu(x)\rangle = \int \frac{d^3\hat{p}}{(2\pi)^3} |h(\hat{p})|^2 \Big(p^\mu - e\langle\alpha|A^\mu(\eta)|\alpha\rangle + \frac{e^2g^2}{kp + e^2g^2}[(p\varepsilon)\varepsilon^{*\mu} + (p\varepsilon^*)\varepsilon^\mu] \\ + k^\mu \left(\frac{(e\langle\alpha|A^\mu(\eta)|\alpha\rangle)}{kp} + \frac{e^2g^2}{kp}(|\alpha|^2 + \tfrac{1}{2} + \tfrac{1}{2}s) - \frac{e^2g^2(e^2g^2 + 2kp)}{(kp + e^2g^2)kp}(\varepsilon p)(\varepsilon^* p)\right) \\ \times \frac{p_0}{kp} \bar{\chi}_s k \chi_s\end{aligned} \tag{53}$$

where $\eta = (1 + e^2g^2/kp)kx$. Expression (53) becomes equal to the expression (50) in the limit of large occupation numbers $|\alpha|^2 \to \infty$. (Factor $g^2|\alpha|^2$ connected with the density of photons is kept constant in this limiting procedure.) The square of $\langle \pi^\mu \rangle$ averaged over time is equal to

$$\overline{\langle \pi(x)\rangle^2} = m^2 + 2e^2g^2(|\alpha|^2 + \tfrac{1}{2} + \tfrac{1}{2}s) = m_*^2. \tag{54}$$

For the spinless particle satisfying the Klein-Gordon equation one obtains identical results if neglects spin terms in (53) and (54).

We note here that for $e^2g^2+kp=0$ all the formulae given above are not valid, this is a reflection of the difficulties of the one-particle theory. For certain quantum numbers a negative frequency solution describes a positron which accelerates even in the vacuum field. This trouble arises from the charge asymmetry of the proposed equations. One can try to compare exact equations of quantum electrodynamics for operators ψ, A with single particle equations. It can be easily shown that the matrix element of operator ψ in the single particle state satisfies the wave equation if one ignores the current in the equation for operator A. This is a good approximation for highly occupied photon states but it violates the commutation relation

$$[A(x), \psi(x')]_{x_0=x_0'}=0 \tag{55}$$

which is necessary for the charge symmetry of the problem. It is possible to improve the theory (Fedorov and Kozakov 1973). We impose requirement (55) and rewrite the term $A(x)\psi$ appearing in the wave equation in its symmetric form $\frac{1}{2}(A\psi+\psi A)$. However, for this case equations become operator-like in character. We found solutions of improved equations. They are really charge symmetric and will be reported elsewhere.

5. Summary

We have presented the solutions of the Dirac equation for the electron interacting with the one-mode circularly polarised electromagnetic field. The wavefunctions obtained differ in form from Volkov states. They are eigenfunctions of the Pauli–Lubanski spin operator. The orthogonality of states has been shown explicitly for the first time. We have found the connection between the quantum numbers and the expectation value of the kinetic momentum. This connection allow us to interpret the vector momentum p which labels the solutions.

Although the results discussed above have been obtained in the case of the one-mode field they can be generalised to an arbitrary quantised plane wave.

References

Backer W and Mitter H 1974 *J. Phys. A: Math. Gen.* **7** 1266

Beers B and Nickle H H 1972 *J. Math. Phys.* **13** 1952

Bergou J and Ehlotzky F 1983 *Phys. Rev.* A **27** 2291

Bergou J and Varro S 1980 *J. Phys. A: Math. Gen.* **14** 2281

Berson I 1969 *Sov. Phys.-JETP* **29** 871

Bialynicki-Birula I 1980 *Foundations of Radiation Theory and Quantum Electrodynamics* ed A O Barut (New York: Plenum)

Bieresteckij W B, Lifshitz E M and Pitajevskij L P 1968 *Relativistskaja Kvantovaja Tieoria* (Moscow: Nauka)

Brown L S and Kibble T W B 1964 *Phys. Rev.* A **133** 705

Brown L S and Kowalski K L 1983 *Phys. Rev. Lett.* **51** 2335

Chakrabarti A 1968 *Nuovo Cimento* **56A** 603

Eberly J H 1969 *Prog. Opt.* **VII** 359

Fedorov M V and Kozakov A E 1973 *Z. Phys.* **261** 191

Filipowicz P 1980 *Bull. Pol. Acad. Sci.* **28** 79

Fried Z, Baker A and Korff D 1966 *Phys. Rev.* **151** 1040

Landau L D and Lifshitz 1973 *Tieoria Polia* (Moscow: Nauka)

Lubanski J K 1942 *Physica* **9** 310

Neville R A and Rörlich F 1971 *Nuovo Cimento* A **1** 625

—— 1971 *Phys. Rev.* D **3** 1962

Redmond P J 1965 *J. Math. Phys.* **151** 1058
Ritus V I 1979 *Trudy FIAN* **111** 5
Sengupta N D 1949 *Bull. Math. Soc. Calcutta* **44** 175
Volkov D M 1935 *Z. Phys.* **94** 250
Von Ross O 1966 *Phys. Rev.* **150** 1112